Minimization, Convolution and Interpolation on Rotation and Motion Groups

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The Basic Idea

In many applications, data about the motion of an object, rather than the object itself, is a quantity requiring analysis or interpolation.

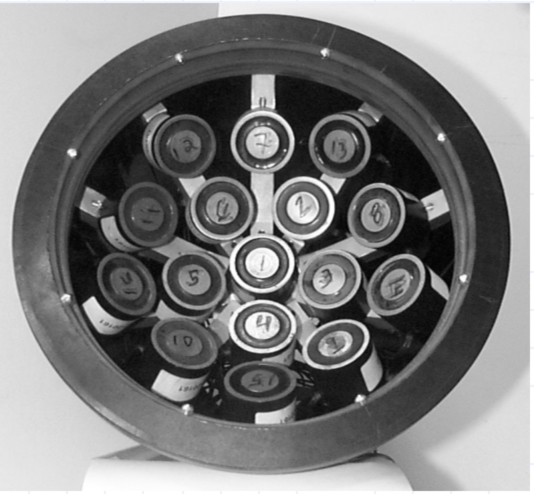
Examples include (but are not limited to):

Spherical Motors
Medical Image Registration
Robotic Manipulators
Protein Motions

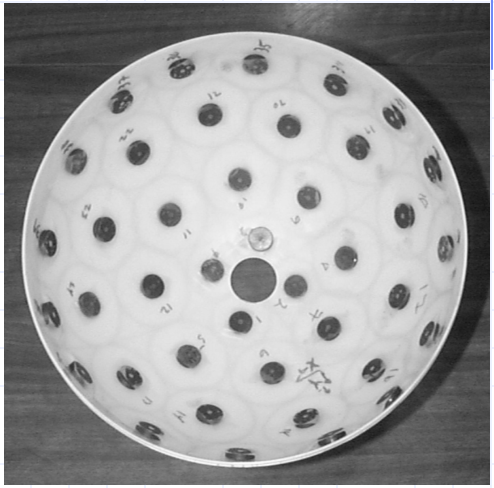
The Spherical Motor

With David Stein and Ed Scheinerman

Our Prototype



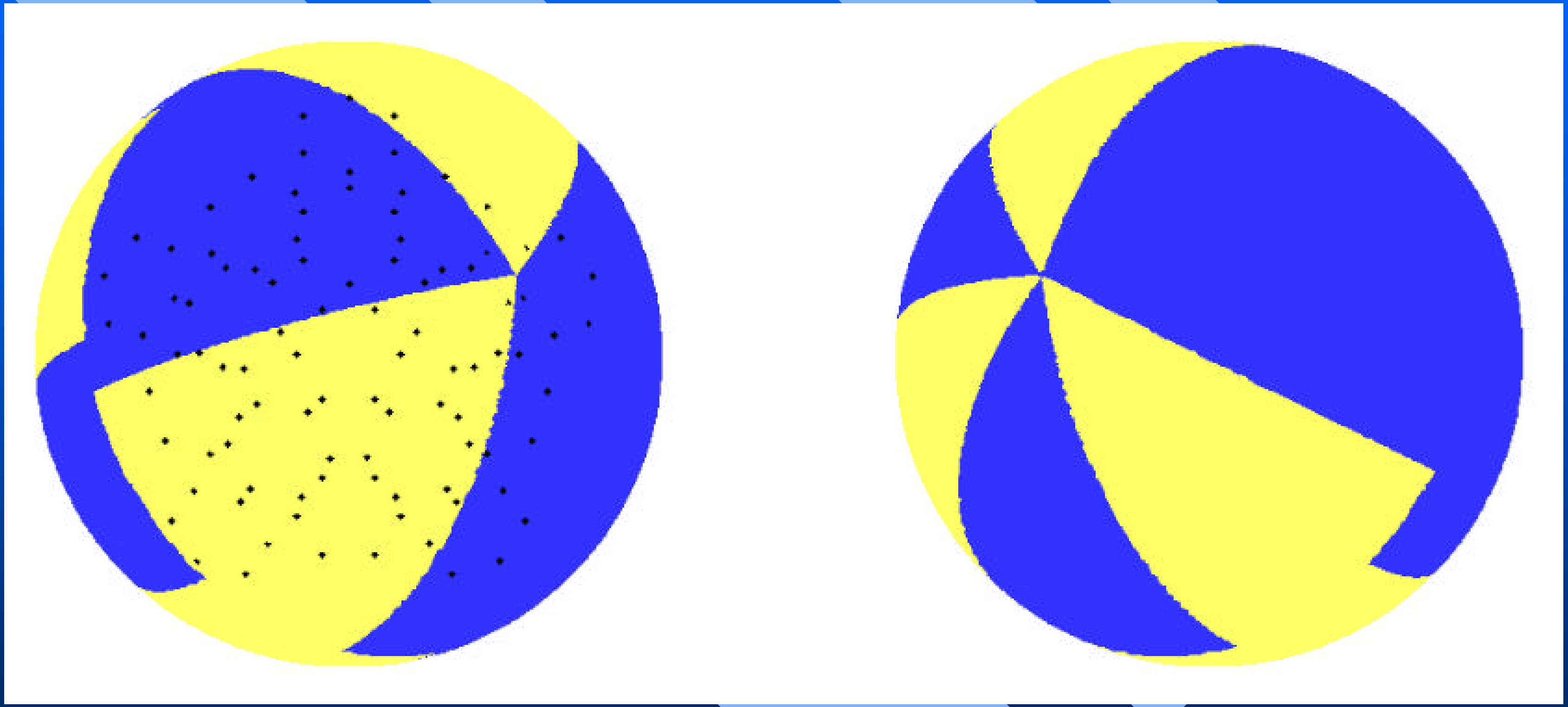
Stator



Rotor

Two Compatible Circle Packings Required

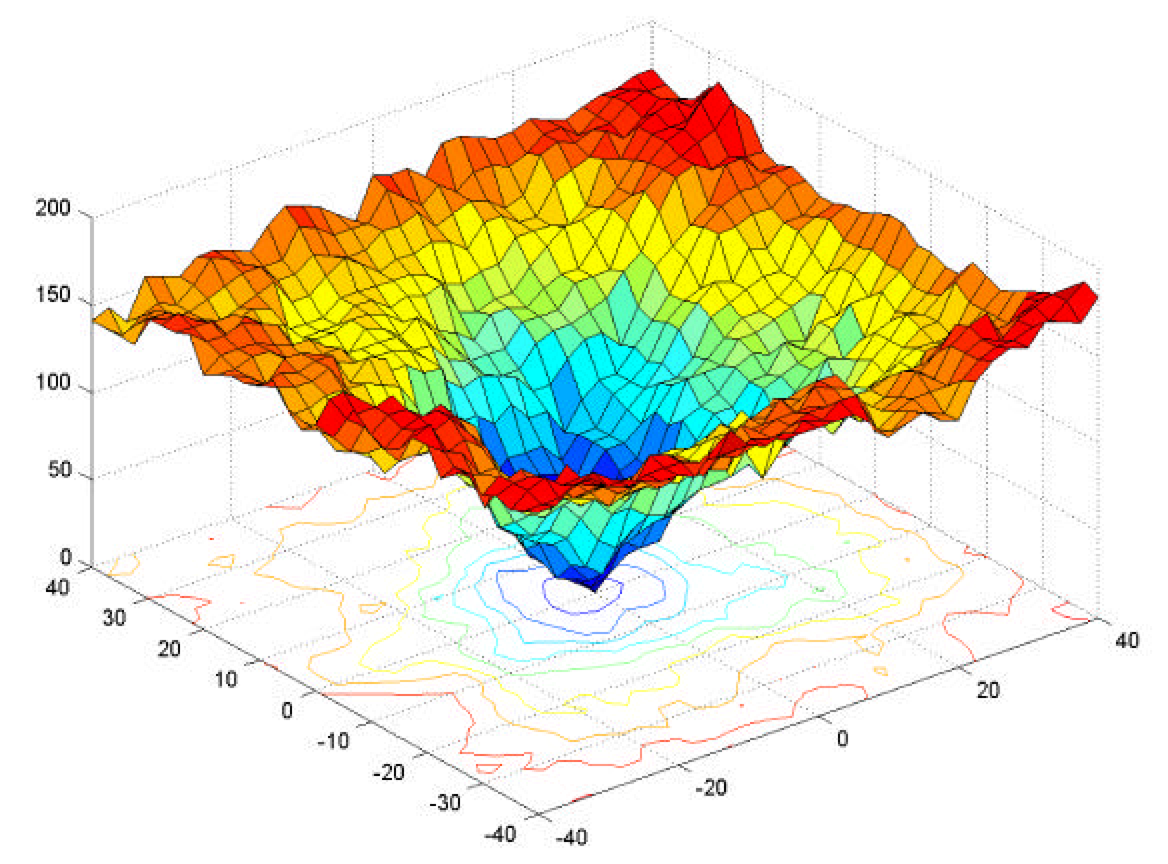
Operating Principles of the Spherical Encoder



$$f(A) = \|\text{measured} - \text{model}(A)\|$$

where A is in SO(3)

Encoder Gives Orientation by Minimization over the Group SO(3)



Local Solution

- Because the domain of *f* is *SO(3)*, we use concepts from Lie groups/algebras to define the gradient of *f*.
- Let **X** be a 3 x 3 skew-symmetric matrix, set:

$$\left(X^R f\right)(A) = \left.\frac{df\left(Ae^{tX}\right)}{dt}\right|_{t=0}$$

- $\mathbf{X}^{\mathbf{R}}$ can be thought of as the right directional derivative of *f* in the direction **X**. Define the right gradient of f at B to be the vector

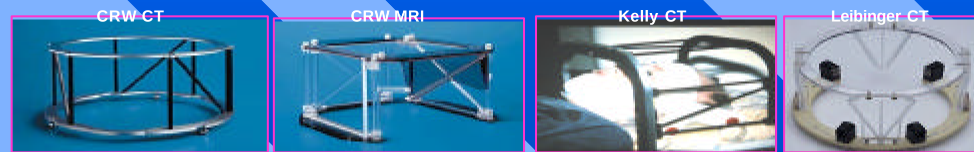
[$E_1^R f$ $E_2^R f$ $E_3^R f$] where,

$$E_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \quad E_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \quad E_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

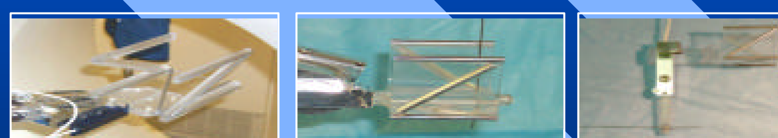
Registration of Fiducials in CT

With Gabor Fichtinger, Sangyoon Lee, Russ Taylor

Problem: Given a stereotactic head cage and a pattern of dots generated by intersecting a plane with the cage, register the image in 3D based on the planar pattern.



Some of the frequently used stereotactic fiducial frames

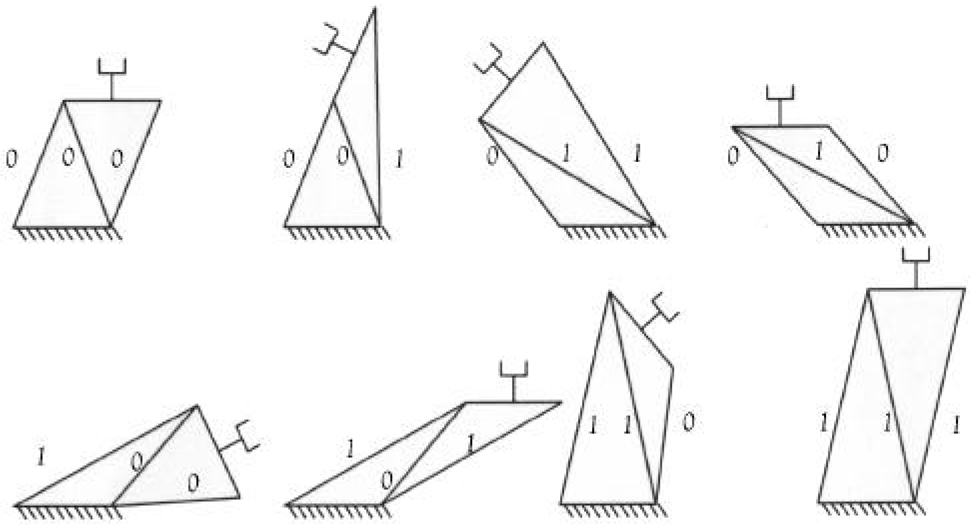


Stereotactic localizers mounted on robotic needle drivers

Incomplete data

Manipulator Workspaces

A Discrete-State Platform Manipulator

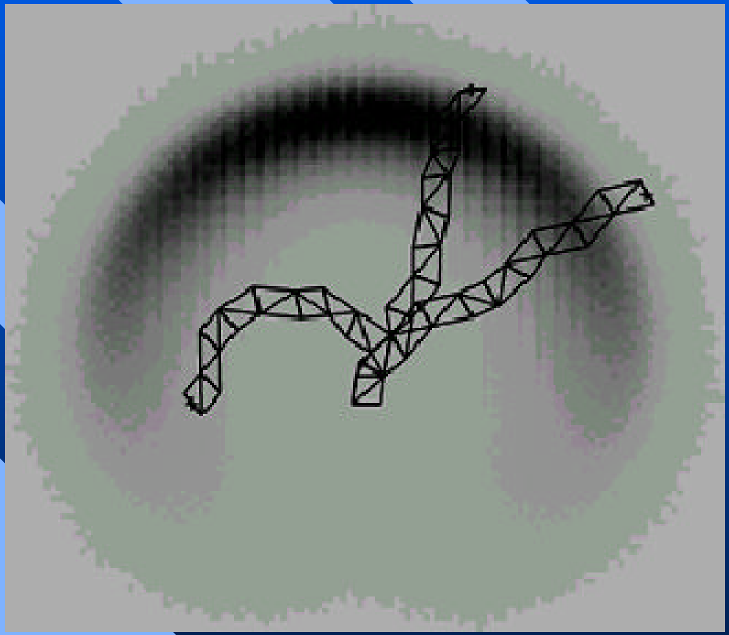


0: retracted state; 1: extended state

Workspace Density

With Imme Ebert-Uphoff

- It describes the density of reachable frames in the workspace.
- It is a probabilistic measurement of accuracy over the workspace.
- The amount of data grows exponentially in the number of platforms



Workspace Generation by Convolution on SE(N)

$$f(g, L_1 + L_2) = \int_G f(h, L_1) f(h^{-1} \mathrm{o}\, g, L_2) dh$$

Planar

$$\int_G dg = \int_0^{2p} \int_0^{2p} \int_0^{\infty} r dr d f d q$$

Spatial

$$\int_G dg = \int_0^{2p} \int_0^{p} \int_0^{2p} \int_0^{p} \int_0^{2p} \int_0^{\infty} r^2 \sin b \sin q \, dr d f d q d a d b d g$$

Fourier Analysis of Motion

With Alexander Kyatkin and Richard Alterndorfer

- Fourier transform of a function of motion, *f(g)*

$$F(f) = \hat{f}(p) = \int_G f(g) U(g^{-1}, p)\, dg$$

$$F(f_1 * f_2) = \hat{f}_2(p) \hat{f}_1(p)$$

- Inverse Fourier transform of a function of motion

$$F^{-1}(\hat{f}) = f(g) = \int trace\,(\hat{f}(p) U(g, p))\, p^{N-1}\, dp$$

where $g \in SE(N)$, p is a frequency parameter,
$U(g,p)$ is a matrix representation of $SE(N)$, and
dg is a volume element at g.

Protein Motions

With Robert Jernigan, Moonki Kim, Adam Schuyler

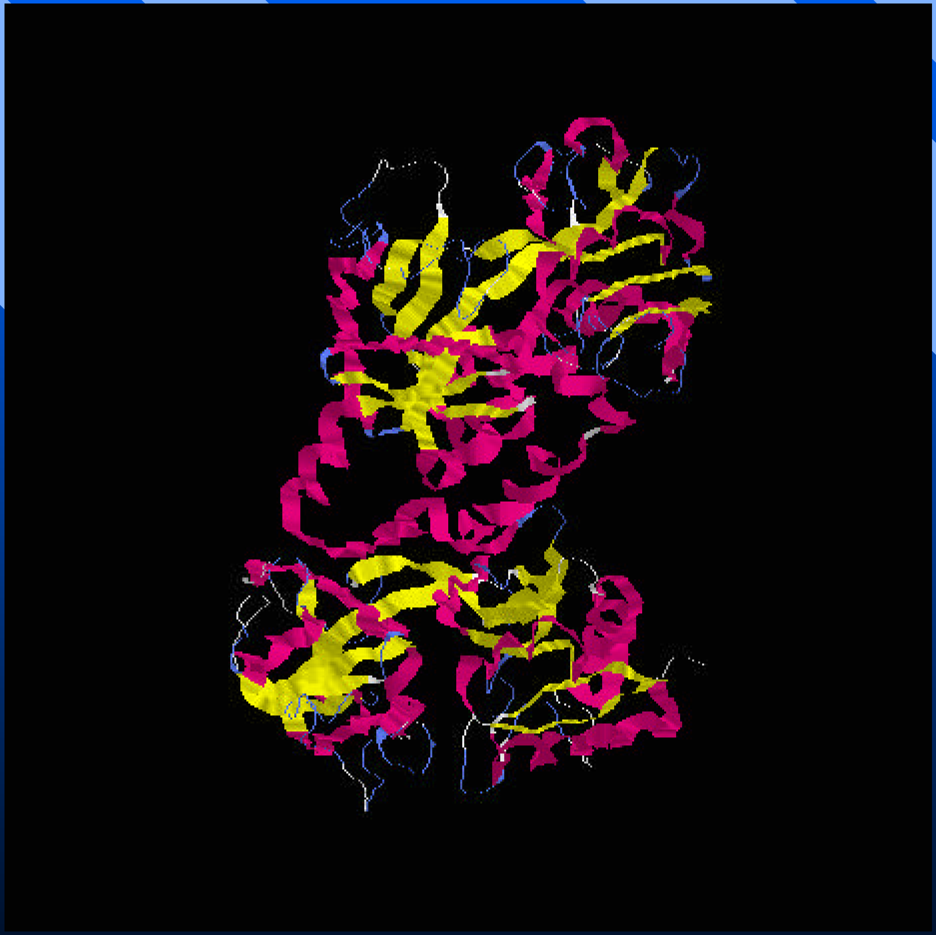
The Basic Idea

Often one protein is crystallized in multiple conformations.

Each conformation is rich in frame information.

We seek to interpolate between two clouds of frames using left-invariant metrics on SE(3).

Lactoferrin Transition from 1lfg.pdb to 1lfh.pdb



Conclusions

Functions of motion arise in a number of disciplines (including Robotics, Biophysics, and Medical Image Registration).

Some classical (20th century) mathematical tools exist for the analysis of data in Lie groups.

Not enough people know about these tools, and few have tried to merge this pure mathematics with computer algorithms and applications.